Incompressible Euler as a limit of complex fluid models with Navier boundary conditions

Adriana Valentina Busuioc, Dragos Iftimie, Milton Lopes Filho, Helena Nussenzveig Lopes

INCOMPRESSIBLE EULER AS A LIMIT OF COMPLEX FLUID MODELS WITH NAVIER BOUNDARY CONDITIONS

A. V. BUSUIOC, D. IFTIMIE, M. C. LOPES FILHO AND H. J. NUSSENZVEIG LOPES

Abstract. In this article we study the limit $\alpha \to 0$ of solutions of the α-Euler equations and the limit $\alpha, \nu \to 0$ of solutions of the second grade fluid equations in a bounded domain, both in two and in three space dimensions. We prove that solutions of the complex fluid models converge to solutions of the incompressible Euler equations in a bounded domain with Navier boundary conditions, under the hypothesis that there exists a uniform time of existence for the approximations, independent of α and ν. This additional hypothesis is not necessary in 2D, where global existence is known, and for axisymmetric flows without swirl, for which we prove global existence. Our conclusion is strong convergence in L^2 to a solution of the incompressible Euler equations, assuming smooth initial data.

1. Introduction

The second grade fluid equations are a model for viscoelastic fluid flow depending on two parameters: the elastic response α and the viscosity ν. When $\nu = 0$ this system is called the Lagrangian-averaged, or α-Euler equations. The main purpose of this article is to study the limiting behavior of solutions of these systems when the parameter α vanishes both for $\nu = 0$ and for $\nu \to 0$, in the case of flows in a bounded domain in $\mathbb{R}^n$, $n = 2, 3$ with Navier boundary conditions. We will prove that, if a weak solution $u^{\alpha,\nu}$ is assumed to exist for a time independent of α and ν, then the limit $\lim_{\alpha,\nu\to 0} u^{\alpha,\nu}$ exists and satisfies the incompressible Euler equations. Global in time existence is known for two dimensional flows, see [6]. In addition, we include three other results in our analysis:

a) global in time existence for axisymmetric flows without swirl, both for $\nu > 0$ and $\nu = 0$, adapting the work [7] to the case of Navier conditions,
b) equivalence of the perfect slip Navier boundary conditions when written using the tangential stress or in terms of the symmetric part of Du.

The second grade fluid equations were introduced by J. E. Dunn and R. L. Fosdick, see [8], as the simplest examples of non-Newtonian fluids of differential type. For viscoelastic fluids one expects the stress tensor to possess memory, or, in other words, to depend on the history of the flow. In fluids of differential type, it is assumed that this memory only applies to the infinitesimal past, so that the stress depends on time derivatives of the flow velocity.

If u is the velocity and p is the scalar pressure of a fluid in motion, the stress tensor for the second grade fluid model is given by

$$\mathbb{S} = -p\mathbb{I} + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where

$$A_1 = A_1(u) = \nabla u + (\nabla u)^t, \text{ and}$$
$$A_2 = A_2(u) = (\partial_t + u \cdot \nabla)A_1 + (\nabla u)^t A_1 + A_1 \nabla u.$$

Here ∇u denotes the Jacobian matrix of u, $(\nabla u)_{i,j} = \partial_{x_j} u^i$. To simplify the notation, we will denote A_1 by A.

We restrict our analysis to the cases $\alpha_1 + \alpha_2 = 0$, $\alpha_1 \geqslant 0$. The physical validity of differential fluid models in general, and of the last assumption in particular, are the subject of controversy in rheology, see [9] and references therein. However, the second grade fluid equations, with these hypothesis on the α_i are a very simple model, mathematically interesting, with a large current literature, see for example [1, 2, 15, 21] with, at least, potential applicability in non-Newtonian fluid modeling. We denote in what follows $\alpha = \alpha_1$, so that the second grade fluid stress tensor takes the form:

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha A^2 + \alpha(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A \nabla u). \tag{1}$$

The α-Euler equations came about in a different way, initially proposed as a desingularization of the incompressible 3D Euler equations with deep geometric significance and relevance in turbulence modeling, see [11]. For some of the recent work concerning the α-Euler equations, see [10, 12, 18, 19].

We are concerned with the limit $\alpha \to 0$, both for $\nu = 0$ and $\nu \to 0$, in the bounded domain case, i.e. where the fluid occupies a bounded, smooth region $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. The limit $\alpha \to 0$ for $\nu > 0$ fixed, was studied by D. Iftimie in [13], and the result obtained is also conditional on finding a uniform time of existence for the approximations. The limit $\alpha \to 0$ for α-Euler was first considered by Bardos, Linshiz and Titi for the vortex sheet problem in 2D and then by Linshiz and Titi in 3D in [3, 17], both in the absence of boundaries. It was also shown in [17] that the time of existence of smooth solutions of α-Euler can be taken independent of α. The main purpose of the present article is to establish a baseline for the study of the limit $\alpha \to 0$ in the presence of boundaries. In this context, it is natural to consider $\nu = 0$ and $\nu \to 0$ at the same time, because the problems are technically very similar.

In order to study the limit $\alpha \to 0$ in domains with boundary, we must supplement the basic evolution equations with boundary conditions. It is natural to consider first the no-slip case $u = 0$ at $\partial\Omega$, but this is not technically within reach, due to the formation of boundary layers. It is well-known, however, that replacing the no-slip condition by the Navier boundary condition for the classical Navier-Stokes equations makes the vanishing viscosity limit analytically treatable, see [4, 14, 20, 23]. That makes considering first problems with the Navier friction condition, at least mathematically, natural. The α-Euler equations have no precise physical meaning as a model, and therefore it is not clear what would be the physically meaningful boundary conditions. For viscoelastic fluids, the occurrence of wall slip is a well documented experimental phenomenon, see [16] and references therein, specifically with *thresholding*, where slip occurs once tangential stresses at the boundary exceed some critical magnitude. In a small viscosity regime, the formation of a boundary layer implies large tangential stresses along the boundary, which makes it natural to consider boundary slip for the study of this limit. In this article, we focus on the special case of perfect slip Navier conditions, where the surface shear stress is assumed to vanish.

We must compare the results obtained here with those obtained by Linshiz and Titi in [17]. They prove convergence from α-Euler solutions to Euler solutions in 3D up to the time of existence of the Euler solution, in the initial data norm, which means strong convergence in H^5. Their proof is based on an idea of N. Masmoudi and it is rather involved. Our main result is convergence of α-Euler and of second grade fluids to Euler in a bounded domain with Navier boundary conditions, with initial data in H^3, assuming the existence of a weak solution in H^1 for a time which is independent of α. We obtain convergence of the solutions in L^2-norm, not strong convergence in H^3. The uniform time of existence is guaranteed for 2D by a result by Busuioc and Ratiu, see [6], and for axisymmetric flow without swirl by an adaptation of another result by the same authors, see [7]. The difficulty in proving the uniformity in α of the time of existence is due to the presence of the boundary. To illustrate that, we include the proof of a result in full space where we prove convergence in L^2 for some time independent of α (smaller than the blow-up time for the limit solution) for initial data in H^5. This is a weaker version of the result in [17], but the proof is much simpler.

The remainder of this paper is divided into five sections. In Section 2, we study two different formulations of the Navier boundary conditions for α-Euler and for second-grade fluids and we show that one of them implies the other. In Section 3 we propose a new weak formulation of the α-Euler and the second grade fluid equations with velocity in H^1. We prove our main convergence result in the Section 4. In Section 5 we prove global existence of solutions for the axisymmetric flow equations without swirl. In Section 6 we add a remark concerning full space flow, and list a few open problems and conclusions.

2. The Navier friction condition

There are two natural ways of extending the Navier friction conditions to the complex fluid models under consideration in the present work: by using the same mathematical condition as in the Newtonian case, which gives a *linear* boundary condition, or by formulating it in terms of the shear stress at the boundary, using (1), which gives a rather complicated nonlinear boundary condition. In the perfect slip case (see below), these two boundary conditions turn out to be largely equivalent. The proof of this fact is the subject of the present section.

The Navier boundary conditions, first introduced by Navier himself, see [22], consist of assuming that the velocity is tangent to the boundary and that the tangential component of the surface velocity is proportional to the surface shear stress at the boundary, i.e.,

$$(\mathbb{S}\hat{n} + \gamma u)_{tan} = 0, \tag{2}$$

where $\hat{n}$ denotes the unit exterior normal to Ω and where the subscript *tan* refers to the tangential component at the boundary.

In this article, we focus on the special case $\gamma = 0$, which is called "perfect slip" Navier boundary condition. Condition 2 is still rather complicated, and we find it more convenient to work with the much simpler condition:

$$(A\hat{n})_{tan} = 0, \tag{3}$$

at the boundary, which is the Newtonian Navier condition. Previous work on second grade fluids with Navier conditions has used (3) as the boundary condition, instead of (2), see [6].

In the following result we will show that (3) is equivalent to (2) with $\gamma = 0$, at least for smooth solutions.

Theorem 1. *Let $u \in C^\infty([0,T];\overline{\Omega})$ be a divergence free vector field satisfying $u \cdot \hat{n} = 0$ on $\partial\Omega$, for every $t \geqslant 0$.*

If

$$(A\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega \tag{4}$$

then

$$(S\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega \tag{5}$$

The converse also holds if we require, in addition to (5), that $[A(u_0)\hat{n}]_{tan} = 0$ on $\partial\Omega$ where $u_0(x) = u(0,x)$.

Proof. We extend $\hat{n}$ to a smooth vector field, also denoted by $\hat{n}$, defined on the whole $\overline{\Omega}$. We need to make this extension because the following calculations involve derivatives of $\hat{n}$ (and not only tangential derivatives).

We decompose $A\hat{n}\big|_{\partial\Omega}$ into normal component and tangential component:

$$A\hat{n} = \beta\hat{n} + w \qquad \text{on } [0,T] \times \partial\Omega \tag{6}$$

where w is tangent to $\partial\Omega$ and $\beta : [0,T] \times \partial\Omega \to \mathbb{R}$. Clearly

$$\beta = A\hat{n} \cdot \hat{n} \in C^\infty([0,T] \times \partial\Omega; \mathbb{R})$$

so

$$w = A\hat{n} - (A\hat{n} \cdot \hat{n})\hat{n} \in C^\infty([0,T] \times \partial\Omega).$$

We also have that

$$A^2\hat{n} = \beta^2\hat{n} + \beta w + Aw.$$

We compute now

$$\begin{aligned} S\hat{n} &= -p\hat{n} + \nu A\hat{n} - \alpha A^2\hat{n} + \alpha(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A\nabla u)n \\ &= (-p + \nu\beta - \alpha\beta^2 + \alpha\partial_t\beta)\hat{n} + \nu w - \alpha\beta w - \alpha Aw + \alpha\partial_t w \\ &\qquad + \alpha(u \cdot \nabla A + (\nabla u)^t A + A\nabla u)\hat{n} \end{aligned} \tag{7}$$

Let us first consider the term $[(u\cdot\nabla)A]\hat{n}$. We use that $u\cdot\nabla$ is a tangential derivative to write

$$\begin{aligned}[(u\cdot\nabla)A]\hat{n} &= (u\cdot\nabla)(A\hat{n}) - A(u\cdot\nabla)\hat{n}\\ &= (u\cdot\nabla)(\beta\hat{n}+w) - A(u\cdot\nabla)\hat{n}\\ &= (u\cdot\nabla\beta)\hat{n} + u\cdot\nabla w + (\beta\mathbb{I}-A)[(u\cdot\nabla)\hat{n}].\end{aligned}$$

Next, we consider the term $(\nabla u)^t A\hat{n} + A\nabla u\hat{n}$. We find:

$$\begin{aligned}(\nabla u)^t A\hat{n} + A\nabla u\hat{n} &= (\nabla u)^t(\beta\hat{n}+w) + A(A-(\nabla u)^t)\hat{n}\\ &= (\nabla u)^t w + \beta w + Aw + \beta^2\hat{n} + (\beta\mathbb{I}-A)[(\nabla u)^t\hat{n}].\end{aligned}$$

so that

$$\begin{aligned}(u\cdot\nabla A + (\nabla u)^t A + A\nabla u)\hat{n} &= (u\cdot\nabla\beta+\beta^2)\hat{n}\\ &\quad + u\cdot\nabla w + (\nabla u)^t w + \beta w + Aw + (\beta\mathbb{I}-A)[(u\cdot\nabla)\hat{n} + (\nabla u)^t\hat{n}].\end{aligned}\tag{8}$$

We identify $(\nabla u)^t\hat{n}$ by examining its components:

$$[(\nabla u)^t\hat{n}]_i = \sum_j(\partial_i u_j)\hat{n}_j = \sum_j \partial_i(u_j\hat{n}_j) - \sum_j u_j\partial_i\hat{n}_j,$$

so that

$$(u\cdot\nabla)\hat{n} + (\nabla u)^t\hat{n} = \nabla(u\cdot\hat{n}) + (u\cdot\nabla)\hat{n} - \sum_j u_j\nabla\hat{n}_j.$$

We recall a result established in [5, Lemma 3], namely that $\sum_j u_j\nabla\hat{n}_j - (u\cdot\nabla)\hat{n}$ is normal to the boundary of Ω whenever u is tangent to the same boundary. Furthermore, since $u\cdot\hat{n}=0$ on $\partial\Omega$, we also have that $\nabla(u\cdot\hat{n})$ is normal to the boundary of Ω. We conclude that there exists some $\delta:[0,T]\times\partial\Omega\to\mathbb{R}$ such that

$$(u\cdot\nabla)\hat{n} + (\nabla u)^t\hat{n} = \delta\hat{n}.\tag{9}$$

Clearly

$$\delta = \delta\hat{n}\cdot\hat{n} = (u\cdot\nabla)\hat{n}\cdot\hat{n} + (\nabla u)^t\hat{n}\cdot\hat{n} = \frac{1}{2}u\cdot\nabla(|\hat{n}|^2) + (\nabla u)^t\hat{n}\cdot\hat{n} = (\nabla u)^t\hat{n}\cdot\hat{n}.$$

From (6) and (9) we obtain that

$$(\beta\mathbb{I}-A)[(u\cdot\nabla)\hat{n} + (\nabla u)^t\hat{n}] = \delta(\beta\mathbb{I}-A)\hat{n} = -\delta w$$

Using this relation in (8) and plugging the result in (7) results in

$$\begin{aligned}S\hat{n} = (-p + \nu\beta - \alpha\beta^2 + \alpha\partial_t\beta + \alpha u\cdot\nabla\beta + \alpha\beta^2)\hat{n} + \nu w&\\ + \alpha\partial_t w + \alpha u\cdot\nabla w + \alpha(\nabla u)^t w - \alpha\delta w&.\end{aligned}$$

We conclude that

$$\frac{1}{\alpha}(S\hat{n})_{tan} = \partial_t w + (\frac{\nu}{\alpha} - \delta)w + [u\cdot\nabla w + (\nabla u)^t w]_{tan} \qquad \text{on } [0,T]\times\partial\Omega.\tag{10}$$

If (4) holds true then $w\equiv 0$, so (10) implies (5).

Conversely, suppose that (5) holds true. Then from (10) we get that

$$\partial_t w + (\frac{\nu}{\alpha} - \delta)w + [u \cdot \nabla w + (\nabla u)^t w]_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega.$$

We multiply this relation by w and integrate on $\partial\Omega$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} (\delta - \frac{\nu}{\alpha})|w|^2 - \int_{\partial\Omega} (\nabla u)^t w \cdot w \leqslant (\frac{\nu}{\alpha} + C\|\nabla u\|_{L^\infty(\partial\Omega)})\|w\|^2_{L^2(\partial\Omega)}.$$

Given that $w_0 = 0$, the Gronwall lemma implies that $w \equiv 0$, that is (4). This completes the proof of the lemma. □

Remark 2. One of the uses of Theorem 1 is to convert solutions of a boundary value problem satisfying condition (4) to one which satisfies condition (5). This is relevant because known global existence results, see [6, 7], assume condition (4). This discussion raises a natural question, as follows. Using Theorem 1, we only know existence of smooth solutions to the second grade fluid equations, or to the α-Euler equations with condition (5) if the initial data satisfies (4), a somewhat unnatural hypothesis. In fact, the space of divergence-free vector fields, tangent to the boundary and satisfying the Newtonian friction condition (4) is dense in the space of all divergence-free vector fields tangent to the boundary in suitable topologies (see [20]). However, if one tries to remove this unnatural boundary condition on the initial data by approximation, we do not obtain enough estimates to prove that the limiting flow satisfies the complex fluid equations. Therefore, existence of a smooth solution to the second-grade or α-Euler equations in a bounded domain, satisfying (5), with smooth initial data that does not satisfy (4), is an interesting open problem.

3. Weak formulation for H^1 solutions

Our next step is to present a weak formulation of the complex fluid models which requires only H^1 spatial regularity for the weak solutions, and incorporates the nonlinear boundary conditions (5) in a natural way.

First let us fix additional notation. Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a smooth, bounded, simply connected domain. For two matrices $M = (m_{ij})$ and $N = (n_{ij})$ we define the dot product $M : N \equiv \sum_{i,j} m_{ij} n_{ij}$. The divergence of a matrix is the vector of the divergences of the rows.

The complex fluid models under consideration take the form:

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{11}$$

where u is the fluid velocity, u_0 is the initial velocity, p is the pressure and $\mathbb{S}$ is the second grade fluid stress tensor, given by (1). We will assume (5) throughout this section. We insert the expression for the stress tensor $\mathbb{S}$ and expand, so that the conservation of momentum equation becomes:

$$\partial_t(u - \alpha\Delta u) + u \cdot \nabla(u - \alpha\Delta u) + \sum_j (u_j - \alpha\Delta u_j)\nabla u_j = -\nabla p + \nu\Delta u.$$

We give now a weak formulation for H^1 solutions of the second grade fluid equations. Assume for the moment that u is sufficiently smooth and let us do some formal calculations. Let φ be a sufficiently regular vector field which is divergence free and tangent to the boundary. We multiply the first line of (11) by φ, integrate in space and time and use an integration by parts to obtain

$$\begin{aligned}(12)\qquad -\int_0^t\int_\Omega u\cdot\partial_t\varphi + \int_\Omega u(t)\cdot\varphi(t) - \int_\Omega u_0\cdot\varphi_0 + \int_0^t\int_\Omega u\cdot\nabla u\cdot\varphi \\ = \int_0^t\int_\Omega \operatorname{div}\mathbb{S}\cdot\varphi = -\int_0^t\int_\Omega \mathbb{S}:\nabla\varphi + \int_0^t\int_{\partial\Omega}\mathbb{S}\hat{n}\cdot\varphi = -\frac{1}{2}\int_0^t\int_\Omega \mathbb{S}:A(\varphi).\end{aligned}$$

We used above the boundary condition (5) and the fact φ is tangent to the boundary to deduce that $\mathbb{S}\hat{n}\cdot\varphi$ vanishes on the boundary. We also used that $\mathbb{S}$ is a symmetric matrix to write that $\mathbb{S}:\nabla\varphi = \frac{1}{2}\mathbb{S}:A(\varphi)$.

The stress tensor is given by

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha A^2 + \alpha\partial_t A + \alpha u\cdot\nabla A + \alpha(\nabla u)^t A + \alpha A\nabla u.$$

Replacing in (12) this formula for the stress tensor and performing a couple of integrations by parts we find

$$\begin{aligned}-\int_0^t\int_\Omega [u\cdot\partial_t\varphi + \frac{\alpha}{2}A:\partial_t A(\varphi)] + \int_\Omega [u(t)\cdot\varphi(t) + \frac{\alpha}{2}A(t)\cdot A(\varphi(t))] - \int_\Omega [u_0\cdot\varphi_0 + \frac{\alpha}{2}A(u_0)\cdot A(\varphi_0)] \\ (13)\qquad + \int_0^t\int_\Omega u\cdot\nabla u\cdot\varphi + \frac{\nu}{2}\int_0^t\int_\Omega A:A(\varphi) - \frac{\alpha}{2}\int_0^t\int_\Omega A^2:A(\varphi) \\ - \frac{\alpha}{2}\int_0^t\int_\Omega u\cdot\nabla A(\varphi):A + \frac{\alpha}{2}\int_0^t\int_\Omega [(\nabla u)^t A + A\nabla u]:A(\varphi) = 0,\end{aligned}$$

for all times t. We used above that u is divergence free and tangent to the boundary to perform an integration by parts in the term $u\cdot\nabla A$.

We wish to use (13), for arbitrary smooth test vector fields φ which are divergence free and tangent to the boundary, as a weak formulation for the equation (11) with boundary condition (5). As with any new weak formulation, we should verify consistency, i.e., that any smooth solution of (11) verifying the boundary condition (5) satisfies the weak formulation (13) and, conversely, any smooth vector field u, divergence free, tangent to the boundary, satisfying (13), will solve (11) with the perfect slip boundary condition (5).

Now, the calculations leading to (13) already show that any smooth solution of (11) verifying the perfect slip Navier boundary conditions (5) satisfies the variational formulation given in (13) for arbitrary test vector fields.

To verify the converse, let u be a smooth vector field which is divergence free, tangent to the boundary and assume that u satisfies (13) for any smooth vector field φ which is divergence free and tangent to the boundary. Then u verifies (11) as well as the perfect slip

Navier boundary conditions (5). Indeed, choosing first φ to be compactly supported in Ω we get (11). Next, consider a test vector field, denoted again by φ, which is not necessarily compactly supported in Ω. Multiply the first equation in (11) by φ and integrate by parts in time, using the initial data for u, to obtain the first equality in (12). Integrating by parts in space we deduce the second equality in (12). Now, equation (13) expresses precisely the equality between the first and last terms in (12) (and hence the third equality in (12)), which can only hold if the boundary term

$$\int_0^t \int_{\partial\Omega} \mathbb{S}\hat{n} \cdot \varphi$$

vanishes for any smooth vector field φ which is divergence free and tangent to the boundary. This implies that the tangential part of $\mathbb{S}\hat{n}$ must vanish, *i.e.* we get (5). This concludes the proof of consistency.

The notion of weak solution we will propose is inspired on the Leray weak solutions for the incompressible Navier-Stokes equations, which includes an energy inequality. As motivation for the appropriate energy inequality in our setting we observe that choosing $\varphi = u$ in (13) yields the following H^1 a priori estimate

$$\int_\Omega (|u|^2 + \frac{\alpha}{2}|A|^2)(t) + \nu \int_0^t \int_\Omega |A|^2 = \int_\Omega (|u_0|^2 + \frac{\alpha}{2}|A(u_0)|^2). \tag{14}$$

We used above the identity $[A(B+C)] : A = (AB) : A + (CA) : A$ that holds true for any symmetric matrix A and arbitrary matrices B and C to deduce that $A^2 : A = [(\nabla u)^t A + A\nabla u] : A$.

In view of the discussion above we introduce the following definition of a weak H^1 solution of the second grade fluid equations.

Definition 3. *We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0,T]$ if and only if*

a) $u \in C^0_w([0,T]; H^1(\Omega))$;
b) *u is divergence free and tangent to the boundary;*
c) *relation* (13) *holds true for all vector fields $\varphi \in C^0([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega))$ which are divergence free and tangent to the boundary;*
d) *the following energy inequality holds true*

$$\int_\Omega (|u|^2 + \frac{\alpha}{2}|A|^2)(t) + \nu \int_0^t \int_\Omega |A|^2 \leqslant \int_\Omega (|u_0|^2 + \frac{\alpha}{2}|A(u_0)|^2) \qquad \forall t \in [0,T]. \tag{15}$$

Remark 4. The properties listed in the previous definition are automatically verified for solutions obtained with a standard approximating procedure, like for instance the Galerkin approximation. This is obvious for b) and c). In fact, relation (13) is stated in general only for $t = T$. But if it is true for $t = T$, then it must hold true for all $t \in [0,T]$. This can be shown by taking $\varphi\chi_{[0,t]}$ as test function after a mollification in time and a passage to the limit. To show a), we observe that from the a priori estimate (14) and using the Korn inequality, the sequence of approximating solutions is bounded in $L^\infty(0,T; H^1(\Omega))$. So the limit solution must belong to $L^\infty(0,T; H^1(\Omega))$. But from the equation we obtain an estimate for $\partial_t u$ implying that $u \in C^0([0,T];\mathcal{D}'(\Omega))$. By density of smooth functions in $L^2(\Omega)$, we have that $L^\infty(0,T;H^1(\Omega)) \cap C^0([0,T];\mathcal{D}'(\Omega)) \subset C^0_w([0,T];H^1(\Omega))$ so we get a). Finally, to get the energy inequality we proceed in the following manner. Denoting by u_n the approximating solution, one has (14) with u replaced by u_n:

$$\int_\Omega (|u_n|^2 + \frac{\alpha}{2}|A(u_n)|^2)(t) + \nu \int_0^t \int_\Omega |A(u_n)|^2 = \int_\Omega (|u_n(0)|^2 + \frac{\alpha}{2}|A(u_n(0))|^2).$$

(with possibly just an inequality instead of an equality, depending on the method of approximation). In the process of passing to the limit, one uses time derivative estimates to obtain equicontinuity, and therefore uniform convergence, in time with values in some negative local Sobolev space. In particular one has that, for all $t \in [0,T]$, $u_n(t) \to u(t)$ in $\mathcal{D}'(\Omega)$. But $u_n(t)$ is bounded in $H^1(\Omega)$, so it possesses a subsequence weakly convergent in $H^1(\Omega)$. By uniqueness of limits in $\mathcal{D}'(\Omega)$, we have that $u_n(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega)$. We can now deduce (15) from the weak lower semicontinuity of the L^2-norm.

4. Limiting behavior when α and ν vanish

It is well-known that, for sufficiently smooth initial velocities u_0, there exists a unique (smooth) solution of the incompressible Euler equations up to some non-zero time $T > 0$. We assume, more precisely, that $u_0 \in H^3(\Omega)$ is divergence-free and tangent to $\partial\Omega$. Then there exists $T > 0$ and a unique velocity $\overline{u} = \overline{u}(x,t) \in C^0([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ that solves the incompressible Euler equations with initial velocity u_0.

We are now ready to state and prove our convergence result.

Theorem 5. *Let $u_0 \in H^3(\Omega)$ be a divergence-free vector field which is tangent to $\partial\Omega$ and which satisfies the boundary condition* (4)*. Let $T > 0$ be a time during which the incompressible Euler equations are well-posed with this initial data.*

Suppose, additionally, that there exists $u^{\nu,\alpha}$ a H^1 weak solution of (11) *with perfect slip Navier boundary conditions in the sense of Definition 3 with initial data u_0, up to time T. Then*

$$\lim_{\nu,\alpha\to 0} \|u^{\nu,\alpha} - \overline{u}\|_{L^\infty(0,T;L^2(\Omega))} = 0.$$

Proof. We can assume without loss of generality that $\alpha \leqslant 1$. During this proof, we denote by C a constant which does not depend on ν and α. The proof is performed through energy estimates. Set

$$w = u^{\nu,\alpha} - \overline{u}.$$

To simplify notation, we will write u instead of $u^{\nu,\alpha}$ and $\overline{A}$ for $A(\overline{u})$. Moreover, we will write

$$\|w\|^2_{H^1_\alpha(\Omega)} = \int_\Omega (|w|^2 + \frac{\alpha}{2}|A(w)|^2).$$

By the Korn inequality, $\|w\|_{H^1_\alpha(\Omega)}$ is equivalent to $\|w\|_{H^1(\Omega)}$ (with constants depending on α). Moreover, $\|w\|_{H^1(\Omega)} \leqslant C\alpha^{-\frac{1}{2}}\|w\|_{H^1_\alpha(\Omega)}$ with C independent of α.

Formally, the following equation holds for w:

$$\partial_t w + (w\cdot\nabla)w + (w\cdot\nabla)\overline{u} + (\overline{u}\cdot\nabla)w = \operatorname{div}(\mathbb{S}) + \nabla\overline{p}, \tag{16}$$

where $\overline{p}$ is the pressure associated to the Euler solution. We would like to multiply (16) by w and integrate over Ω. However, w is only H^1 so we don't have enough regularity to perform this. Nevertheless, it is still possible to get the desired conclusion using the variational formulation (13) coupled with the energy inequality (15). Indeed, multiplying the equation of w by w is, at least at the formal level, equivalent to multiplying the equation of u by $\overline{u}$, adding the result to the equation of $\overline{u}$ multiplied by u and subsequently subtracting the result of this addition from the equation of $\overline{u}$ multiplied by $\overline{u}$ added to the equation of u multiplied by u. All these operations are permitted except for the multiplication of the equation of u by u. But the multiplication of the equation of u by u results in the energy equality and we can use the energy inequality instead; this results in an inequality instead of an equality at the end.

More precisely, the rigorous argument is the following. We multiply the Euler equation satisfied by $\overline{u}$ with u to obtain

$$\int_0^t\int_\Omega \partial_t\overline{u}\cdot u+\int_0^t\int_\Omega \overline{u}\cdot\nabla\overline{u}\cdot u=0. \tag{17}$$

We recall next the conservation of energy that holds true for the Euler equation:

$$\|\overline{u}(t)\|^2_{L^2(\Omega}=\|u_0\|^2_{L^2(\Omega}. \tag{18}$$

We add (18) to the energy inequality (15) and subtract (17) multiplied by 2 and (13) multiplied by 2 and with φ replaced by $\overline{u}$. We obtain after some straightforward calculations the following inequality

$$\begin{aligned}\int_\Omega(|w(t)|^2+&\frac{\alpha}{2}|A(w)|^2)+\nu\int_0^t\int_\Omega|A-\frac{\overline{A}}{2}|^2\\ \leqslant&\frac{\alpha}{2}\int_\Omega(|\overline{A}|^2-|A(u_0)|^2)-\alpha\int_0^t\int_\Omega A:\partial_t\overline{A}+\frac{\nu}{4}\int_0^t\int_\Omega|\overline{A}|^2\\ &+2\int_0^t\int_\Omega u\cdot\nabla u\cdot\overline{u}+2\int_0^t\int_\Omega\overline{u}\cdot\nabla\overline{u}\cdot u-\alpha\int_0^t\int_\Omega(u\cdot\nabla\overline{A}):A\\ &-\alpha\int_0^t\int_\Omega A^2:\overline{A}+\alpha\int_0^t\int_\Omega[(\nabla u)^tA+A\nabla u]:\overline{A}.\end{aligned} \tag{19}$$

We continue by estimating the second line of (19):

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega(|\overline{A}|^2-&|A(u_0)|^2)-\alpha\int_0^t\int_\Omega A:\partial_t\overline{A}+\frac{\nu}{4}\int_0^t\int_\Omega|\overline{A}|^2\\ \leqslant&C\alpha\big(\|\overline{u}\|^2_{L^\infty(0,t;H^1(\Omega))}+\|u_0\|^2_{H^1(\Omega)}+t\|A\|_{L^\infty(0,t;L^2(\Omega))}\|\partial_t\overline{A}\|_{L^\infty(0,t;L^2(\Omega))}\big)\\ &\qquad+C\nu t\|\overline{u}\|^2_{L^\infty(0,t;H^1(\Omega))}\\ \leqslant&C\alpha\big(\|\overline{u}\|^2_{L^\infty(0,T;H^1(\Omega))}+\|u_0\|^2_{H^1(\Omega)}\big)+C\alpha^{\frac12}T\|u_0\|_{H^1(\Omega)}\|\overline{u}\|_{C^1([0,T];H^1(\Omega))}\\ &\qquad+C\nu t\|\overline{u}\|^2_{L^\infty(0,T;H^1(\Omega))}.\end{aligned} \tag{20}$$

where we used the energy inequality (15) to deduce the last line.

Next, we make an integration by parts and write

$$
\begin{aligned}
(21)\quad 2\int_0^t\int_\Omega u\cdot\nabla u\cdot\overline{u}+2\int_0^t\int_\Omega \overline{u}\cdot\nabla\overline{u}\cdot u &= -2\int_0^t\int_\Omega w\cdot\nabla\overline{u}\cdot u\\
&= -2\int_0^t\int_\Omega w\cdot\nabla\overline{u}\cdot w\leqslant C\int_0^t\|w\|^2_{L^2(\Omega)}\|\nabla\overline{u}\|_{L^\infty(\Omega)}\\
&\leqslant C\|\overline{u}\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t\|w\|^2_{H^1_\alpha(\Omega)}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
-\alpha\int_0^t\int_\Omega(u\cdot\nabla\overline{A}):A=&-\alpha\int_0^t\int_\Omega(u\cdot\nabla\overline{A}):A(w)\\
\leqslant& C\alpha\int_0^t\|u\|_{H^1(\Omega)}\|\nabla\overline{A}\|_{H^1(\Omega)}\|A(w)\|_{L^2(\Omega)}\\
(22)\qquad\leqslant& C\alpha\int_0^t(\|\overline{u}\|_{H^1(\Omega)}+\|w\|_{H^1(\Omega)})\|\nabla\overline{A}\|_{H^1(\Omega)}\|A(w)\|_{L^2(\Omega)}\\
\leqslant& C\alpha\int_0^t\|\overline{u}\|^2_{H^3(\Omega)}\|A(w)\|_{L^2(\Omega)}+C\|\overline{u}\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t\|w\|^2_{H^1_\alpha(\Omega)}\\
\leqslant& C\alpha t\|\overline{u}\|^3_{L^\infty(0,T;H^3(\Omega))}+C\|\overline{u}\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t\|w\|^2_{H^1_\alpha(\Omega)}
\end{aligned}
$$

We bound now the last line in (19). We observe that

$$
[(\nabla u)^tA+A\nabla u]:\overline{A}-A^2:\overline{A}=[(\nabla u)^tA+A\nabla u-A^2-(\nabla\overline{u})^t\overline{A}-\overline{A}\nabla\overline{u}+\overline{A}^2]:\overline{A}
$$

so

$$
|[(\nabla u)^tA+A\nabla u]:\overline{A}-A^2:\overline{A}|\leqslant C|\nabla\overline{u}||\nabla w|(|\nabla\overline{u}|+|\nabla w|)
$$

We infer the following bound for the last line in (19)

$$
\begin{aligned}
-\alpha\int_0^t\int_\Omega A^2:\overline{A}+\alpha\int_0^t\int_\Omega[(\nabla u)^tA+A\nabla u]:\overline{A}&\\
(23)\qquad\leqslant& C\alpha\int_0^t\int_\Omega|\nabla\overline{u}||\nabla w|(|\nabla\overline{u}|+|\nabla w|)\\
\leqslant& C\|\nabla\overline{u}\|_{L^\infty([0,T]\times\Omega)}\int_0^t\|w\|^2_{H^1_\alpha(\Omega)}+C\alpha t\|\nabla\overline{u}\|^3_{L^\infty([0,T]\times\Omega)}
\end{aligned}
$$

Using estimates (20)–(23) in (19) implies that

$$
\begin{aligned}
(24)\quad \|w(t)\|^2_{H^1_\alpha(\Omega)}\leqslant{}& C\alpha\big(\|\overline{u}\|^2_{L^\infty(0,T;H^1(\Omega))}+\|u_0\|^2_{H^1(\Omega)}\big)+C\alpha^{\frac12}T\|u_0\|_{H^1(\Omega)}\|\overline{u}\|_{C^1([0,T];H^1(\Omega))}\\
&+C\nu T\|\overline{u}\|^2_{L^\infty(0,T;H^1(\Omega))}+C\alpha T\|\overline{u}\|^3_{L^\infty(0,T;H^3(\Omega))}+C\|\overline{u}\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t\|w\|^2_{H^1_\alpha(\Omega)}.
\end{aligned}
$$

Applying the Gronwall lemma to (24) implies that

$$\lim_{\nu,\alpha\to 0} \|w\|_{L^\infty(0,T;H^1_\alpha(\Omega))} = 0.$$

This completes the proof of Theorem 5. $\square$

A first corollary of Theorem 5, together with Theorem 1 and Theorem 1 in [6] is convergence of the vanishing viscosity limit in the two-dimensional case.

5. Axisymmetric flow without swirl

Another situation where the uniform time of existence of solutions of the second-grade fluid equations may be established, and we can conclude convergence of the vanishing viscosity limit is for axisymmetric flows without swirl. Global-in-time existence of smooth solutions for the viscous equation was established in [7] with Dirichlet boundary conditions. For the present work, we require the same result with Navier friction conditions, which should actually be easier to prove, but it is not available in the literature. The purpose of this section is to outline an adaptation of Theorem 1 in [7] to the case of Navier friction conditions.

Let Ω be a bounded smooth axisymmetric domain of $\mathbb{R}^3$ with axis of rotation $\mathbb{R}(0,0,1)$. A flow is said to be axisymmetric if the velocity has cylindrical symmetry:

$$u(t,x) = a(t,r,x_3)(x_1,x_2,0) + b(t,r,x_3)(x_2,-x_1,0) + c(t,r,x_3)(0,0,1) \equiv u_r + u_\theta + u_3 \tag{25}$$

where a, b, c are scalar functions and $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$.

The flow is said axisymmetric without swirl if the swirl velocity u_θ vanishes. Due to the invariance by rotation of the second grade fluids equations, the special structure expressed in (25) is preserved by the flow. We will show below that an axisymmetric second grade fluid verifying the perfect slip Navier boundary conditions preserves the no swirl condition. We have the following proposition.

Proposition 6. *Suppose that u_0 is axisymmetric without swirl, belongs to $H^3(\Omega)$, is divergence free and tangent to the boundary and verifies the boundary conditions* (4)*. Suppose in addition that $\frac{1}{r}\operatorname{curl}(u_0 - \alpha\Delta u_0) \in L^2(\Omega)$. Then there exists a global H^3 no swirl solution.*

Proof. As was proved in [6], there exists a local H^3 solution. If this solution blows up in finite time, then the $H^3(\Omega)$ norm must become infinite. We will show that this cannot happen, so that the solution is global.

We show first that the swirl velocity vanishes. We observe first that u_θ is divergence free and tangent to the boundary. Indeed, the vector field $(x_2, -x_1, 0)$ is tangent to the boundary and any vector field of the form $f(r, x_3)(x_2, -x_1, 0)$ is divergence free. We show now that u_θ verifies the perfect slip Navier boundary conditions (4).

Computing $A\hat{n}$ using relation (25) shows after some straightforward calculations that the swirl component of $A\hat{n}$ is exactly $A(u_\theta)\hat{n}$ (they are both equal to $(\hat{n}\cdot\nabla b)(x_2 - x_1, 0)$). Given that the swirl component is always tangent to the boundary, we infer from (4) that we must have that

$$A(u_\theta)\hat{n} = 0$$

at the boundary. In particular, the swirl velocity must verify the perfect slip Navier boundary conditions (4).

We recall next the second grade fluid equation can be written under the form

$$\partial_t v - \nu\Delta u + u\cdot\nabla v + \sum_j v_j \nabla u_j = -\nabla p \tag{26}$$

where $v = u - \alpha\Delta u$. We multiply the above equation by the swirl velocity and integrate in space to get

$$\int_\Omega \partial_t v\cdot u_\theta - \nu\int_\Omega \Delta u\cdot u_\theta + \int_\Omega u\cdot\nabla v\cdot u_\theta + \int_\Omega \sum_j v_j\nabla u_j\cdot u_\theta = 0.$$

Recall next the following identity: if u is a divergence free tangent to the boundary vector field that verifies the Navier boundary conditions (4) and w is a vector field tangent to the boundary, then we have that

$$\int_\Omega \Delta u\cdot w = -\frac{1}{2}\int_\Omega A(u) : A(w). \tag{27}$$

It is easy to check that the swirl component of Δu is Δu_θ. Using also relation (27) we can write

$$\begin{aligned}\int_\Omega \partial_t v\cdot u_\theta - \nu\int_\Omega \Delta u\cdot u_\theta &= \int_\Omega \partial_t(u_\theta - \alpha\triangle u_\theta)\cdot u_\theta - \nu\int_\Omega \Delta u_\theta\cdot u_\theta\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|u_\theta|^2 + \frac{\alpha}{2}|A(u_\theta)|^2) + \frac{\nu}{2}\int_\Omega |A(u_\theta)|^2.\end{aligned}$$

Next, we make an integration by parts to write

$$\int_\Omega u\cdot\nabla v\cdot u_\theta + \int_\Omega \sum_j v_j\nabla u_j\cdot u_\theta = \int_\Omega (u_\theta\cdot\nabla u - u\cdot\nabla u_\theta)\cdot v$$

A straightforward calculation using relation (25) shows that the vector field $u_\theta\cdot\nabla u - u\cdot\nabla u_\theta$ is a multiple of $(x_2, -x_1, 0)$. More precisely,

$$u_\theta\cdot\nabla u - u\cdot\nabla u_\theta = -u\cdot\nabla b(x_2, -x_1, 0).$$

We infer that

$$\begin{aligned}\int_\Omega (u_\theta\cdot\nabla u - u\cdot\nabla u_\theta)\cdot v &= \int_\Omega (u_\theta\cdot\nabla u - u\cdot\nabla u_\theta)\cdot v_\theta\\ &= \int_\Omega (u_\theta\cdot\nabla u - u\cdot\nabla u_\theta)\cdot(u_\theta - \alpha\Delta u_\theta)\\ &= \int_\Omega u_\theta\cdot\nabla u\cdot u_\theta - \alpha\int_\Omega (u_\theta\cdot\nabla u - u\cdot\nabla u_\theta)\cdot\Delta u_\theta.\end{aligned}$$

We estimate now

$$\Big|\int_\Omega u_\theta\cdot\nabla u\cdot u_\theta\Big| \leqslant \|u_\theta\|^2_{L^2(\Omega)}\|\nabla u\|_{L^\infty(\Omega)} \leqslant C\|u_\theta\|^2_{L^2(\Omega)}\|u\|_{H^3(\Omega)}.$$

We use next relation (27) and the fact that the vector field $u_\theta \cdot \nabla u - u \cdot \nabla u_\theta$ is tangent to the boundary to write

$$\int_\Omega (u_\theta \cdot \nabla u - u \cdot \nabla u_\theta) \cdot \Delta u_\theta = -\frac{1}{2} \int_\Omega A(u_\theta \cdot \nabla u - u \cdot \nabla u_\theta) : A(u_\theta). \tag{28}$$

When expanding $A(u_\theta \cdot \nabla u - u \cdot \nabla u_\theta)$ we get the terms $u_\theta \cdot \nabla A(u) - u \cdot \nabla A(u_\theta)$ plus other terms which are of the following form: a first order derivative of u multiplied by a first order derivative of u_θ. Putting these other terms back in (28) we see that they can be bounded by $C\|\nabla u\|_{L^\infty(\Omega)} \|\nabla u_\theta\|^2_{L^2(\Omega)}$. We infer that we can bound

$$\begin{aligned}\alpha \int_\Omega (u_\theta \cdot \nabla u - u \cdot \nabla u_\theta) \cdot \Delta u_\theta \leqslant -\frac{\alpha}{2} \int_\Omega u_\theta \cdot \nabla A(u) : A(u_\theta) + \frac{\alpha}{2} \int_\Omega u \cdot \nabla A(u_\theta) : A(u_\theta) \\ + C\|u_\theta\|^2_{H^1(\Omega)} \|u\|_{H^3(\Omega)}.\end{aligned}$$

The middle term on the right-hand side above vanishes. We bound the first one as follows:

$$\begin{aligned}-\frac{\alpha}{2} \int_\Omega u_\theta \cdot \nabla A(u) \cdot A(u_\theta) &\leqslant \frac{\alpha}{2} \|u_\theta\|_{L^2(\Omega)} \|\nabla A(u)\|_{L^4(\Omega)} \|u_\theta\|_{L^4(\Omega)} \\ &\leqslant C\|u_\theta\|_{L^2(\Omega)} \|\nabla A(u)\|_{H^1(\Omega)} \|u_\theta\|_{H^1(\Omega)} \\ &\leqslant C\|u_\theta\|_{L^2(\Omega)} \|u\|_{H^3(\Omega)} \|u_\theta\|_{H^1_\alpha(\Omega)}\end{aligned}$$

Putting together all these relations we obtain in the end the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_\theta\|^2_{H^1_\alpha(\Omega)} + \nu \int_\Omega |A(u_\theta)|^2 \leqslant C(\alpha) \|u_\theta\|^2_{H^1_\alpha(\Omega)} \|u\|_{H^3(\Omega)}.$$

Since u_θ vanishes at the initial time, the Gronwall inequality implies that u_θ vanishes for all times. This completes the proof that the swirl velocity vanishes.

Now that we know that the velocity is axisymmetric without swirl, the proof continues in the same manner as the proof of [7, Theorem 1] as presented on pages 110–112. The only difference is that in [7] the authors consider Dirichlet boundary conditions, while here we deal with perfect slip Navier boundary conditions. Fortunately, the required tools to work with perfect slip Navier boundary conditions are available in [6]. We shall only sketch the proof, highlighting the places where the perfect slip Navier boundary conditions come into play.

Let $\omega = \operatorname{curl} u$. For axisymmetric flows without swirl, one has that

$$\omega = \tilde{\omega}(t, r, x_3)(x_2, -x_1, 0) \quad \text{and} \quad \omega - \alpha \triangle \omega = \breve{\omega}(t, r, x_3)(x_2, -x_1, 0)$$

where $\tilde{\omega}, \breve{\omega}$ are scalar functions. As observed in [7], $\breve{\omega}$ verifies the following equation:

$$\partial_t \breve{\omega} + \frac{\nu}{\alpha}(\breve{\omega} - \tilde{\omega}) + u \cdot \nabla \breve{\omega} = 0.$$

By hypothesis, we have that $\breve{\omega}(0) \in L^2(\Omega)$. Performing L^2 estimates and using that u is tangent to the boundary implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\breve{\omega}\|^2_{L^2(\Omega)} + \frac{\nu}{\alpha} \|\breve{\omega}\|^2_{L^2(\Omega)} \leqslant \frac{\nu}{\alpha} \|\tilde{\omega}\|^2_{L^2(\Omega)} \leqslant C\|u\|^2_{H^2(\Omega)},$$

so that

$$\sup_{[0,t]} \|\check{\omega}\|^2_{L^2(\Omega)} \leqslant C + C \sup_{[0,t]} \|u\|^2_{H^2(\Omega)}$$

Using the regularity result proved in [6, Proposition 6] we have that

$$\|u\|_{H^3(\Omega)} \leqslant C\|\omega - \alpha\Delta\omega\|_{L^2(\Omega)} + C\|u\|_{H^1(\Omega)}.$$

By the H^1 energy estimates, the quantity $\|u\|_{H^1(\Omega)}$ is uniformly bounded in time. Moreover, since the domain is bounded we also have that $\|\omega - \alpha\Delta\omega\|_{L^2(\Omega)} \leqslant C\|\check{\omega}\|_{L^2(\Omega)}$. We infer that

$$\sup_{[0,t]} \|u\|^2_{H^3(\Omega)} \leqslant C + C \sup_{[0,t]} \|u\|^2_{H^2(\Omega)} \leqslant C + C \sup_{[0,t]} \|u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}.$$

Given that $\|u\|_{H^1(\Omega)}$ is uniformly bounded the above relation implies a bound for $\|u(t)\|_{H^3(\Omega)}$ for all times. This completes the proof. □

6. Conclusions

There are several natural questions that arise from the analysis presented in this article. A first issue is the need to assume a uniform time of existence for the family of weak solutions of the complex flow equations in Theorem 5. In the Newtonian case such a hypothesis is not needed, since we have global existence of Leray solutions. Global-in-time existence of H^1-weak solutions both for the second-grade fluid model and for α-Euler, in $3D$ are natural, and rather interesting open problems. An easier problem in this direction is to obtain a finite time of existence of the weak solutions for the complex fluid models which is independent of α and ν. In fact, such a result is known in the full-space case, see [17]. Below we outline a simple argument which obtains such a uniform time of existence in the full space. The result is weaker than the one in [17], but the proof is much simpler.

Suppose that $u_0 \in H^5(\mathbb{R}^3)$. Let σ be a multi-index of order $\leqslant 3$. We apply ∂^σ to (26), we multiply by $\partial^\sigma v$ and sum over $|\sigma| \leqslant 3$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^3(\mathbb{R}^3)} + \sum_{|\sigma|\leqslant 3} \big(\nu\|\nabla\partial^\sigma u\|^2_{L^2(\Omega)} + \alpha\nu\|\Delta\partial^\sigma u\|^2_{L^2(\Omega)}\big) = -\sum_{|\sigma|\leqslant 3}\Big[\int \partial^\sigma(u\cdot\nabla v)\partial^\sigma v + \sum_j \int \partial^\sigma(v_j\nabla u_j)\partial^\sigma v\Big].$$

We have the classical inequality

$$\Big|\int \partial^\sigma(u\cdot\nabla v)\partial^\sigma v\Big| \leqslant K_1\|u\|_{H^3(\mathbb{R}^3)}\|v\|^2_{H^3(\mathbb{R}^3)}$$

where K_1 is a universal constant. We also have that $\|u\|_{H^3(\mathbb{R}^3)} \leqslant \|v\|_{H^3(\mathbb{R}^3)}$ so

$$\Big|\int \partial^\sigma(u\cdot\nabla v)\partial^\sigma v\Big| \leqslant K_1\|v\|^3_{H^3(\mathbb{R}^3)}.$$

One can show in a similar manner that

$$\Big|\int \partial^\sigma(v_j\nabla u_j)\partial^\sigma v\Big| \leqslant K_2\|v\|^3_{H^3(\mathbb{R}^3)}$$

where K_2 is another universal constant. We deduce from the above relations that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^3(\mathbb{R}^3)} \leqslant K\|v\|^3_{H^3(\mathbb{R}^3)}.$$

where K is a universal constant. This implies that v is bounded in $H^3(\mathbb{R}^3)$ on a time interval that depends only on $\|v_0\|_{H^3(\mathbb{R}^3)} = \|u_0 - \alpha\Delta u_0\|_{H^3(\mathbb{R}^3)}$.

Note that the argument above shows that, in the full plane case, the velocity u is bounded in $H^3(\mathbb{R}^3)$ on a time interval independent of α. We observe that this should not be true in the case of a bounded domain. Indeed, suppose that there is a sequence of solutions uniformly bounded in $H^3(\Omega)$ on a time interval independent of α. Then by Theorem 5 this sequence of solutions converges to the solution of the Euler equation. But given the boundedness in $H^3(\Omega)$, this would imply that the solution of the Euler equation verifies the perfect slip Navier boundary conditions. This is of course not true in general even though the initial velocity verifies these boundary conditions. The question of existence of a solution for a time independent of α, ν in the case of a $3D$ bounded domain with Navier friction condition remains open, and, as we have seen, the answer cannot be obtained by means of a simple energy argument as above.

Another interesting open question is the existence of a solution to the second-grade fluid equations with physical boundary condition (2), if the initial data does not satisfy the nonphysical boundary condition (3). Other natural lines of inquiry include investigating the vanishing viscosity limit in the two-dimensional case, with more irregular initial data and also in the case of threshold slip data of the type studied in [16].

Acknowledgments: M. C. Lopes Filho's research was partially supported by CNPq grant #303.301/2007-4. H. J. Nussenzveig Lopes' research was partially supported by CNPq grant #302.214/2004-6. This work was supported in part by FAPESP project #2007/51490-7 and by the CNRS-FAPESP cooperation project #22076. The authors would like to thank L. J. Nussenzveig Lopes for his assistance during this project.

References

[1] H. Abboud and T. Sayah, *Upwind discretization of a time-dependent two-dimensional grade-two fluid model.* Comput. Math. Appl. **57** (2009), 1249–1264.

[2] I. Ahmad, M. Sajid, T. Hayat, *Heat transfer in unsteady axisymmetric second grade fluid.* Appl. Math. Comput. **215** (2009), 1685-1695.

[3] C. Bardos, J. Linshiz and E. Titi, *Global regularity for a Birkhoff-Rott-α approximation of the dynamics of vortex sheets of the 2D Euler equations.* Phys. D **237** (2008), 1905–1911.

[4] T. Clopeau, A. Mikelic and R. Robert, *On the vanishing viscosity limit for the 2D incompressible Navier– Stokes equations with the friction type boundary conditions.* Nonlinearity **11** (1998), 1625–1636

[5] A. V. Busuioc and D. Iftimie, *A non-Newtonian fluid with Navier boundary conditions.* J. Dynam. Differential Equations 18 (2006), no. 2, 357–379.

[6] A. V. Busuioc and T. Ratiu, *The second grade fluid and averaged Euler equations with Navier-slip boundary conditions.* Nonlinearity, **16** (2003), 1119–1149.

[7] A. V. Busuioc and T. Ratiu, *Some remarks on a certain class of axisymmetric fluids of differential type.* Physica D, **191** (2004), 106–120.

[8] J. E. Dunn and R. L. Fosdick, *Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade.* Arch. Rat. Mech. An. **56** (1974) 191–252.
[9] J. E. Dunn and K. R. Rajagopal, *Fluids of differential type - critical review and thermodynamic analysis.* Int. J. Engr. Sci. **33** (1995), 689–729.
[10] Jishan Fan and Tohro Ozawa, *On the regularity criteria for the generalized Navier-Stokes equations and Lagrangian averaged Euler equations.* Diff. Int. Eqs **21** (2008), 443–457.
[11] D. Holm, J. Marsden and T. Ratiu, *Euler-Poincare models of ideal fluids with nonlinear dispersion.* Phys. Rev. Lett. **80** (1998), 4173–4176.
[12] Thomas Hou and Congming Li, *On global well-posedness of the Lagrangian averaged Euler equations.* SIAM J. Math. Anal. **38** (2006), 782–794.
[13] D. Iftimie, *Remarques sur la limite $\alpha \to 0$ pour les fluides de grade 2.* Studies in Mathematics and its Applications v. 31, D. Cioranescu and J. L. Lions (editors), Elsevier, 2002.
[14] D. Iftimie and G. Planas, *Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions.* Nonlinearity **19** (2006), 899–918.
[15] M. Khan, S. Hyder Ali, H. Qi, *Exact solutions for some oscillating flows of a second grade fluid with a fractional derivative model.* Math. Comput. Model. **49** (2009), 1519-1530.
[16] C. le Roux, *Existence and uniqueness of the flow of second-grade fluids with slip boundary conditions.* Arch. Rat. Mech. Anal. **148** (1999), 309–356.
[17] J. Linshiz and E. Titi, *On the convergence rate of the Euler-α inviscid second-grade complex fluid, model to the Euler equations.* J. Stat. Phys., **138** (2010), 305–332.
[18] Xiaofeng Liu and Houyu Jia, *Local existence and blowup criterion of the Lagrangian averaged Euler equations in Besov spaces.* Commun. Pure Appl. Anal. **7** (2008), 845–852.
[19] Xiaofeng Liu, Meng Wang and Zhifei Zhang, *A note on the blowup criterion of the Lagrangian averaged Euler equations.* Non. Anal. T.M.A. **67** (2007), 2447–2451.
[20] M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas, *On the inviscid limit for 2D incompressible flow with Navier friction condition.* SIAM J. Math. Anal. **36** (2005), 1130–1141.
[21] Mehrdad Massoudi, Phuoc X. Tran, R. Wulandana, *Convection-Radiation Heat Transfer in a Nonlinear Fluid with Temperature-Dependent Viscosity.* Math. Prob. Eng., **2009**, article ID 232670.
[22] C. L. M. H. Navier, *Mémoire sur les lois du mouvement des fluides.* Mémoires de l'Académie Royale des Sciences de l'Institut de France **VI** (1823) 389–440.
[23] Y. Xiao and Z. Xin, *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition.* Comm. Pure Appl. Math. **60** (2007), 1027–1055.

(A.V. Busuioc) Université de Lyon, F-42023 Saint-Etienne, France – Laboratoire de Mathématiques de l'Université de Saint-Etienne – Faculté des Sciences et Techniques – 23 rue Docteur Paul Michelon – 42023 Saint-Etienne Cedex 2, France
E-mail address: valentina.busuioc@univ-st-etienne.fr

(D. Iftimie) Université de Lyon, Université Lyon 1 – CNRS UMR 5208 Institut Camille Jordan – 43 bd. du 11 Novembre 1918 – Villeurbanne Cedex F-69622, France.
E-mail address: iftimie@math.univ-lyon1.fr
URL: http://math.univ-lyon1.fr/~iftimie

(M. C. Lopes Filho) Depto. de Matemática–IMECC, Universidade Estadual de Campinas – UNICAMP, Campinas, SP 13083-970, Brazil
E-mail address: mlopes@ime.unicamp.br

(H. J. Nussenzveig Lopes) Depto. de Matemática–IMECC, Universidade Estadual de Campinas – UNICAMP, Campinas SP 13083-970, Brazil
E-mail address: hlopes@ime.unicamp.br